Transition operators for the free convolution

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1. Introduction

2. Computing conditional expectations
   - The algebra $\mathbb{C}\{X\}$
   - Free convolution operators

3. Free Hall transform
   - Another characterization
   - The large-N limit

Main problem

Let A, B be free random variables in a W^*-probability space $(\mathcal{A}, \tau)$.
There is a unique conditional expectation from $\mathcal{A}$ to $W^*(B)$, denoted by $\tau(\cdot|B)$.
We consider $\tau(P(A+B)|B)$ for any polynomial P.

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Theorem (Biane 1998)

If A and B are self-adjoint, there is a Feller-Markov kernel $k_{A,B}(x, \mathrm{d}y)$ such that, for all Borel bounded function $f : \mathbb{R} \to \mathbb{R}$,

$$\tau(f(A+B)|B) = (K_{A,B}f)(B)$$

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Goal: construct a framework to avoid the self-adjointness, the dependence in B, and the limitation of f to be univariate.

Motivation

Let $t > 0$. Let S_t be a semi-circular variable of variance t in a W^*-probability space $(\mathcal{A}, \tau)$.
Let B be a random variable free from S_t.
We have $\tau((S_t + B)^3 | B) = B^3 + 2tB + t\tau(B)$.

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We guess that there is an abstract object

$$X^3 + 2tX + t\tau(X),$$

which is independent of B. The space of polynomials has to be extended.

Universal property of $\mathbb{C}\{X\}$

The algebra $\mathbb{C}[X]$ possesses the following universal property: for all element A of an algebra $\mathcal{A}$, there exists a unique algebra homomorphism φ such that $\varphi(X) = A$.

$$X \in \mathbb{C}[X] \overset{\varphi}{\dashrightarrow} \mathcal{A} \ni A, \ \varphi(X) = A.$$

Center-valued expectation

$$X \in \mathbb{C}[X] \overset{\varphi}{\dashrightarrow} \mathcal{A} \ni A, \;\; \varphi(X) = A.$$

A center-valued expectation τ is a linear function from $\mathcal{A}$ to its center such that

1. for all $A, B \in \mathcal{A}$, we have $\tau(\tau(A)B) = \tau(A)\tau(B)$;
2. $\tau(1_{\mathcal{A}}) = 1_{\mathcal{A}}$.

Universal property of $\mathbb{C}\{X\}$

There exists an algebra $\mathbb{C}\{X\}$ endowed with a center-valued expectation tr which possesses the following universal property: for all element A of an algebra $\mathcal{A}$ endowed with a center-valued expectation τ, there exists a unique algebra homomorphism φ such that $\varphi(X) = A$ and $\varphi \circ \mathrm{tr} = \tau \circ \varphi$.

$$X \in \mathbb{C}\{X\} \overset{\varphi}{\dashrightarrow} \mathcal{A} \ni A, \ \varphi(X) = A, \ \varphi \circ \mathrm{tr} = \tau \circ \varphi.$$

More about $\mathbb{C}\{X\}$

The space $\mathbb{C}\{X\}$ is unique up to an isomorphism.
We have naturally $\mathbb{C}[X] \subset \mathbb{C}\{X\}$.
Furthermore,

$$\{X^{k_0}\,\mathrm{tr}(X^{k_1})\cdots\mathrm{tr}(X^{k_n}) : n \in \mathbb{N}, k_0, \ldots, k_n \in \mathbb{N}\}$$

is a basis of $\mathbb{C}\{X\}$, called the canonical basis.

The $\mathbb{C}\{X\}$-calculus

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Polynomial calculus: for all $P \in \mathbb{C}[X]$, we set $P(A) = \varphi(P)$.

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$$X \in \underset{\circlearrowleft \, \mathrm{tr}}{\mathbb{C}\{X\}} \overset{\varphi}{\dashrightarrow} \underset{\circlearrowleft \, \tau}{\mathcal{A}} \ni A, \ \varphi(X) = A, \ \varphi \circ \mathrm{tr} = \tau \circ \varphi.$$

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$$\begin{tikzcd} X \in & \mathbb{C}\{X\} \arrow[loop above, "\mathrm{tr}"] \arrow[r, dashed, "\varphi"] & \mathcal{A} \arrow[loop above, "\tau"] & \ni A,\ \ \varphi(X) = A,\ \ \varphi \circ \mathrm{tr} = \tau \circ \varphi. \end{tikzcd}$$

$\mathbb{C}\{X\}$-calculus: for all $P \in \mathbb{C}\{X\}$, we set $P(A) = \varphi(P)$.
For all $n \in \mathbb{N}, k_0, \ldots, k_n \in \mathbb{N}$, if we set $P = X^{k_0}\,\mathrm{tr}(X^{k_1})\cdots\mathrm{tr}(X^{k_n})$, we have

$$P(A) = A^{k_0}\tau(A^{k_1})\cdots\tau(A^{k_n}).$$

Main theorem

Theorem

Let $A \in \mathcal{A}$. There exists an operator Δ_A on $\mathbb{C}\{X\}$ such that, for all $P \in \mathbb{C}\{X\}$, and all $B \in \mathcal{A}$ free from A, we have

$$\tau\left(P\left(A+B\right)\middle|\, B\right) = \left(e^{\Delta_A} P\right)\left(B\right).$$

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Example: we have, for all $B \in \mathcal{A}$ free from S_t,

$$\tau\left(\left(S_t+B\right)^3 \middle|B\right) = \left(e^{\Delta_{S_t}}(X^3)\right)\left(B\right) = B^3 + 2tB + t\tau(B).$$

Other versions:

- There exists also an operator D_A for the multiplicative case: $\tau\left(P\left(AB\right)\middle|\, B\right) = \left(e^{D_A}P\right)\left(B\right)$.
- The multivariate case requires the space $\mathbb{C}\{X_i : i \in I\}$.

Description of Δ_A

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$$\Delta_A P = \sum_{i=0}^{n} X^{k_0}\,\mathrm{tr}(X^{k_1}) \cdots \mathrm{tr}(\Delta_A(X^{k_i})) \cdots \mathrm{tr}(X^{k_n}).$$

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It suffices to describe $\Delta_A(X^n)$ for all $n \in \mathbb{N}$.

Description of Δ_A

Let $n \in \mathbb{N}$.

$$\Delta_A(X^n) = \sum_{1 \leq k_1 < \ldots < k_m \leq n} \kappa_m(A) \cdot \overbrace{X \cdots X}^{k_1 - 1} \mathrm{tr}(\overbrace{X \cdots X}^{k_2 - k_1 - 1}) \cdots \mathrm{tr}(\overbrace{X \cdots X}^{k_m - k_{m-1} - 1}) \overbrace{X \cdots X}^{n - k_m},$$

where $\kappa_m(A)$ $(m \geq 1)$ are the free cumulants of A.

An example: the semi-circular case

Let $t > 0$ and S_t be a semi-circular variable of variance t.
The free cumulants of S_t are $\kappa_1(S_t) = 0$, $\kappa_2(S_t) = t$ and $\kappa_n(S_t) = 0$ for all $n > 2$.
We have
$\Delta_{S_t} X^3 = 2tX + t\,\mathrm{tr}(X)$,
and $(\Delta_{S_t})^2 X^3 = \Delta_{S_t}(2tX + t\,\mathrm{tr}(X)) = 0$.
Thus,

$$e^{\Delta_{S_t}}(X^3) = X^3 + \Delta_{S_t} X^3 + 0 = X^3 + 2tX + t\,\mathrm{tr}(X).$$

Using the theorem, we have, for all $B \in \mathcal{A}$ free from S_t,

$$\tau\left((S_t + B)^3 \,|\, B\right) = (e^{\Delta_{S_t}}(X^3))\,(B) = B^3 + 2tB + t\tau(B).$$

Free multiplicative Brownian motion

The (right) free unitary Brownian motion $(U_t)_{t\geq 0}$ is defined to be the solution of the following free stochastic differential equation

$$\begin{cases} U_0 & = & 1, \\ \mathrm{d}U_t & = & i\,\mathrm{d}S_t U_t - \frac{1}{2} U_t\,\mathrm{d}t. \end{cases}$$

where S_t is a free semicircular process.

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where S_t is a free semicircular process. Similarly, the (right) free circular multiplicative Brownian motion $(G_t)_{t\geq 0}$ is the solution of the free stochastic differential equation

$$\begin{cases} G_0 &= 1, \\ \mathrm{d}G_t &= \mathrm{d}C_t\,G_t. \end{cases}$$

where C_t is a free circular process.

Free Hall transform

We denote by $L^2(U_t, \tau)$ and $L^2_{\mathrm{hol}}(G_t, \tau)$ the Hilbert completion of the algebra generated respectively by U_t and U_t^{-1}, and by G_t and G_t^{-1} (for the norm $\|\cdot\|_2 : A \mapsto \tau(A^*A)^{1/2}$).

Theorem (Biane 1997)

Let $t > 0$. There exists a Hilbert space isomorphism $\mathcal{F}_t$ between $L^2(U_t, \tau)$ and $L^2_{\mathrm{hol}}(G_t, \tau)$, called the free Hall transform.

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Theorem (C 2013)

Let $t>0$. For all $P \in \mathbb{C}[X]$, $\mathcal{F}_t(P(U_t)) = (e^{D_{U_t}}P)(G_t)$. Moreover, if U_t and G_t are free, for all $P \in \mathbb{C}\{X\}$,

$$\mathcal{F}_t\Big(P(U_t)\Big) = \tau\Big(P(U_tG_t)\Big|G_t\Big).$$

Multiplicative Brownian motions

The (right) Brownian motion $(U_t^{(N)})_{t\geq 0}$ on $U(N)$ is defined to be the solution of the following stochastic differential equation

$$\begin{cases} U_0^{(N)} & = & 1, \\ \mathrm{d}U_t^{(N)} & = & i\,\mathrm{d}H_t\,U_t^{(N)} - \frac{1}{2}U_t^{(N)}\,\mathrm{d}t. \end{cases}$$

where H_t is a Hermitian Brownian motion in $M_N(\mathbb{C})$.

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where H_t is a Hermitian Brownian motion in $M_N(\mathbb{C})$. Similarly, the (right) Brownian motion $(G_t^{(N)})_{t\geq 0}$ on $GL_N(\mathbb{C})$ is the solution of the stochastic differential equation

$$\begin{cases} G_0^{(N)} = 1, \\ \mathrm{d}G_t^{(N)} = \mathrm{d}Z_t\,G_t^{(N)}. \end{cases}$$

where Z_t is a complex Brownian motion in $M_N(\mathbb{C})$.

The classical Segal-Bargmann-Hall transform

We denote by ρ_t and μ_t the respective laws of $U_t^{(N)}$ and $G_t^{(N)}$.

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Theorem (Hall 1994)

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We consider $B_t^{(N)} : L^2(\rho_t) \otimes M_N(\mathbb{C}) \to L^2_{\mathrm{hol}}(\mu_t) \otimes M_N(\mathbb{C})$.
The $\mathbb{C}\{X\}$-calculus is adapted in this framework: for all $P \in \mathbb{C}\{X\}$,

$$P = \Big(U \mapsto P(U)\Big) \in L^2(\rho_t) \otimes M_N(\mathbb{C}).$$

The large-N limit

For all $P \in \mathbb{C}\{X\}$, $B_t^{(N)}(P) = e^{\frac{t}{2}\Delta_{U(N)}}P$ and $\mathcal{G}_t(P) = e^{D_{U_t}}P$.

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For all $P \in \mathbb{C}\{X\}$, $B_t^{(N)}(P) = e^{\frac{t}{2}\Delta_{U(N)}} P$ and $\mathcal{G}_t(P) = e^{D_{U_t}} P$.
But the Laplace operator $\Delta_{U(N)}$ satisfies

$$\frac{t}{2}\Delta_{U(N)} = D_{U_t} + O(1/N^2)$$

when acting on the functions given by the $\mathbb{C}\{X\}$-calculus.

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Theorem (C, Driver-Hall-Kemp 2013)

Let $t > 0$. For all $P \in \mathbb{C}[X, X^{-1}]$, we have

$$\left\| B_t^{(N)}(P) - \mathcal{G}_t(P) \right\|^2_{L^2_{\mathrm{hol}}(\mu_t) \otimes M_N(\mathbb{C})} = O(1/N^2).$$

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- related work:
  Todd Kemp. arXiv:1306.2140 and arXiv:1306.6033