Supports of measures in a free additive convolution semigroups

Hao-Wei Huang

Queen's University

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Outline

Free convolution

Given probability measures μ and ν on $\mathbb{R}$, $\mu \boxplus \nu$ is the distribution of $X + Y$, where X and Y are free selfadjoint operators with respective distributions μ and ν.

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- $$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-s}\, d\mu(s), \qquad z \in \mathbb{C}^+,$$

  Cauchy transform of μ is one-to-one in some truncated cone

  $$\Gamma_{\alpha,\beta} = \{x + iy \in \mathbb{C}^+ : y > \alpha,\ |x| \leq \beta y\}.$$

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- Voiculesce transform: $\mathcal{R}_\mu(z) = G_\mu^{-1}(z) - 1/z$, $z \in \Gamma_{\alpha,\beta}$ is a linearizing transform for $\boxplus$, i.e.,

  $$\mathcal{R}_{\mu \boxplus \nu} = \mathcal{R}_\mu + \mathcal{R}_\nu.$$

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- The functions ω_j, $j = 1, 2$, are uniquely determined and satisfy

$$\lim_{y \to +\infty} \frac{\omega_j(iy)}{iy} = 1.$$

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- The existence of $\{\mu^{\boxplus t} : t \geq 1\}$ was proved by Nica and Speicher for μ with compact support.
- Later, Belinschi and Bercovici gave another proof of the existence of the full generalization by using subordination result and Denjoy-Wolff points.

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- continuous function $\omega_t : \mathbb{C}^+ \cup \mathbb{R} \to \mathbb{C}^+ \cup \mathbb{R}$ such that $\omega_t(\mathbb{C}^+) \subset \mathbb{C}^+$, $\omega_t|\mathbb{C}^+$ is analytic, and $H_t(\omega_t(z)) = z$, $z \in \mathbb{C}^+$.

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- We have

$$F_{\mu^{\boxplus t}} = F_\mu \circ \omega_t.$$

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- Density of $\mu^{\boxplus t}$: analytic wherever it is positive.
- $\alpha \in \mathbb{R}$ is an atom of $\mu^{\boxplus t}$ if and only if

$$\mu(\{\alpha/t\}) > 1 - t^{-1},$$

in which case

$$\mu^{\boxplus t}(\{\alpha\}) = t\mu(\{\alpha/t\}) - 1.$$

Let $H_t(z) = tz - (t-1)F_\mu(z)$, $z \in \mathbb{C}^+$, and let $\Omega_t = \{z \in \mathbb{C}^+ : \Im H_t(z) > 0\}$ be as before.

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(4) $\psi_t(x) = H_t(x + if_t(x))$ is a homeomorphism on $\mathbb{R}$.
(5) Density of $(\mu^{\boxplus t})^{\mathrm{ac}}$ is given by

$$\frac{d(\mu^{\boxplus t})^{\mathrm{ac}}}{dx}(\psi_t(x)) = \frac{t(t-1)f_t(x)}{\pi|tx - \psi_t(x) + itf_t(x)|^2} \sim \frac{f_t(x)}{\pi(x^2 + f_t^2(x))}, \quad x \in \mathbb{R}.$$

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(3) The number of the components in $\mathrm{supp}(\mu^{\boxplus t})$ is at most countable for all t and is a non-increasing function of t.

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Example: μ is compactly supported $\Rightarrow n(t) = 1$ for large t.

Counterexample

- For any Borel probability measure ν there exists a unique measure μ with mean zero and unit variance such that

$$F_\mu(z) = z - G_\nu(z), \quad z \in \mathbb{C}^+.$$

$n(t) = \infty$ for all $t > 1$

Consider the measure

$$\nu = \sum_{n=1}^{\infty} 2^{-n} \delta_{2^n}.$$

Then $\mu^{\boxplus t}$ contains infinitely many numbers of components in the support.

(Huang, Zhong, 2013) Similar results hold for the free multiplicative semigroup $\{\mu^{\boxtimes t} : t \geq 1\}$.

Thank You!