Amalgamated Free Products of Hyperfinite W^*-Algebras

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The Amalgamated Free Product

Definition

We say that von Neumann algebras A and B are *free with amalgamation* over a subalgebra D in some larger algebra $\mathcal{M}$ with trace preserving conditional expectation onto D if $E_D(x_1 x_2 \ldots x_n) = 0$ whenever $E_D(x_i) = 0$ for all i and the x_i alternate between A and B.

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Definition

The *amalgamated free product* of two von Neumann algebras A and B over common subalgebra D, $A *_D B$, is a von Neumann algebra generated by A and B so that A and B are free with amalgamation D in $A *_D B$.

Hyperfinite von Neumann Algebras

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Semifinite hyperfinite von Neumann algebras can be written as the direct sum of the following types of algebras:

- Matrix Algebras
- $B(\mathcal{H})$
- Matrix Algebras or $B(\mathcal{H})$ tensor $L^\infty(\nu)$, where ν is diffuse and semifinite.
- $R \otimes L^\infty(\mu)$, where R is the hyperfinite II_1 factor.
- $R \otimes B(\mathcal{H}) \otimes L^\infty(\mu)$.

In general we will be working with semifinite von Neumann algebras, with specified trace, and specified trace preserving expectation onto subalgebra D (where the trace should also be semifinite). In general our morphisms will be trace preserving.

Interpolated Free Group Factors

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They have the following properties:

1. If $r \in \mathbb{Z}, 2 \leq r \leq \infty$ then $L(F_r)$ is the factor associated to the free group on r elements.
2. For $1 < r, r' \leq \infty$, $L(F_r) * L(F_{r'}) = L(F_{r+r'})$.
3. For $1 < r \leq \infty$ and $0 < \gamma < \infty$, $L(F_r)_\gamma = L(F_{1+(r-1)/\gamma^2})$, where $L(F_r)_\gamma$ is the compression or dilation of $L(F_r)$ by γ.

- $\mathcal{R}_1$ (referred to as $\mathcal{R}$ in Dykema's original paper) is the class of finite von Neumann algebras which are the finite direct sum of: Matrix algebras, Matrix algebras tensor $L^\infty([0,1])$, Hyperfinite II_1 factors, and interpolated free group factors.

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- $\mathcal{R}_4$ is the class of semifinite von Neumann algebras which are the direct sum of a hyperfinite von Neumann algebra and a countable number of interpolated free group factors and $B(\mathcal{H}) \otimes L(F_t)$.

Free Dimension

Definition

Let A be a finite von Neumann algebra in $\mathcal{R}_3$, written in the following format

$$A = H \oplus \bigoplus_{i \in I} \overset{p_i}{L(F_{r_i})} \oplus \bigoplus_{j \in J} \underset{t_j}{M_{n_j}},$$

where H is a diffuse hyperfinite algebra, the $L(F_{r_i})$ are interpolated free group factors. The *free dimension*

$$\mathrm{fdim}(A) = 1 + \left(\sum_{i \in I} \tau(p_i)^2 (r_i - 1) \right) - \sum_{j \in J} t_j^2.$$

1. For A a diffuse hyperfinite algebra, $\text{fdim}(A) = 1$.
2. For $A = L(F_r)$, an interpolated free group factor, $\text{fdim}(A) = r$.
3. For $A = \underset{t_j}{\oplus_{j \in J} M_{n_j}}$ a multimatrix algebra,
   $\text{fdim}(A) = 1 - \sum_{j \in J} t_j^2$.
4. For any A, $\text{fdim}(A) \geq 0$, and $\text{fdim}(\mathbb{C}) = 0$.

Hyperfinite von Neumann Algebras over $\mathbb{C}$

Theorem (Dykema 1993)

The standard free product of two finite hyperfinite von Neumann algebras A and B is of the form

$$F \oplus \bigoplus_{i \in I} M_{n_i}$$

where F is an interpolated free group factor or diffuse type I hyperfinite algebra, and I is finite. Furthermore the $fdim(A * B) = fdim(A) + fdim(B)$.

The Amalgamated Free Product of Multimatrix Algebras

Theorem (Dykema 1995)

For A and B multimatrix algebras with subalgebra D, $A *_D B$ is in $\mathcal{R}_3$, and if D is finite dimensional then it is in $\mathcal{R}_2$. Furthermore $fdim(A *_D B) = fdim(A) + fdim(B) - fdim(D)$.

Theorem (Dykema 2011)

$\mathcal{R}_1$ is closed under amalgamated free products over finite dimensional subspaces. They also follow the formula $fdim(A *_D B) = fdim(A) + fdim(B) - fdim(D)$.

Standard Embeddings

Definition

A *standard embedding* is a unital embedding which includes an interpolated free group factor into another, $L(F_t) \to L(F_s)$, by taking a generating set of $L(F_t)$, $R \cup \{p_i X_i p_i\}_{i \in I}$ to a larger generating set for $L(F_s)$, $R \cup \{p_i X_i p_i\}_{i \in I'}$, where $I \subset I'$.

Properties of Standard Embeddings

- For $A = L(F_s)$ and $B = L(F_{s'})$, $s < s'$, then for $\phi : A \to B$ and projection $p \in A$, ϕ is standard if and only if $\phi|_{pAp} \to \phi(p)B\phi(p)$ is standard.

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- The inclusion $A \to A * B$ is standard if A is an interpolated free group factor and B is an interpolated free group factor, $L(\mathbb{Z})$, or a finite dimensional algebra other than $\mathbb{C}$.

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- The inclusion $A \to A * B$ is standard if A is an interpolated free group factor and B is an interpolated free group factor, $L(\mathbb{Z})$, or a finite dimensional algebra other than $\mathbb{C}$.
- The composition of standard embeddings is standard.
- For $A_n = L(F_{s_n})$, with $s_n < s_{n'}$ if $n < n'$, and $\phi_n : A_n \to A_{n+1}$ a sequence of standard embeddings, then the inductive limit of the A_n with the inclusions ϕ_n is $L(F_s)$ where $s = \lim_{n\to\infty} s_n$.

Theorem (Dykema, R. 2011)

Let A and B be finite hyperfinite von Neumann algebras with finite dimensional subalgebra D. Then $A *_D B$ is in $\mathcal{R}_2$. Furthermore $fdim(A *_D B) = fdim(A) + fdim(B) - fdim(D)$.

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Thus we can assume D is abelian, and thus isomorphic to $\bigoplus_{i=1}^{n} \underset{t_n}{\overset{p_n^D}{\mathbb{C}}}$

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- Thus a sequence $\mathcal{M}(i,j) = A_i *_D B_j$ which approximates $\mathcal{M} = A *_D B$.

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- Thus a sequence $\mathcal{M}(i,j) = A_i *_D B_j$ which approximates $\mathcal{M} = A *_D B$.
- Each $\mathcal{M}(i,j)$ is the amalgamated free product of multimatrix algebras over a multimatrix subalgebra.
- Thus we can apply Dykema's result to determine $\mathcal{M}(i,j)$.

We call an embedding a *simple step* if it of the following forms:

1. $\underset{t}{M_n} \oplus A \to \left(\bigoplus_{i=1}^{m} \underset{t_i}{M_n} \right) \oplus A.$
2. $\underset{t}{M_n} \oplus \underset{t}{M_m} \oplus A \to \underset{t}{M_{n+m}} \oplus A$

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Assume each step $A_i \to A_{i+1}$ and $B_j \to B_{j+1}$ is a simple step.

Lemma

For a minimal projection $p \in M_n$ less than some minimal projection in D.

$$p(((M_n \otimes A) \oplus B) *_D C)p \cong p((M_n \oplus B) *_D C)p * A$$

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$$p\left(\left((M_n \otimes A) \oplus B\right) *_D C\right) p \cong p\left((M_n \oplus B) *_D C\right) p * A$$

Proof.

- Note the left is generated by $p\left((M_n \oplus B) *_D C\right) p$ and $A \cong p(M_n \otimes A)p$.

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$$p\,(((M_n \otimes A) \oplus B) *_D C)\,p \cong p\,((M_n \oplus B) *_D C)\,p * A$$

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Thus we can assume that if $A_i \longrightarrow A_{i+1}$ is a simple step of the first kind then $\mathcal{M}(i,j) \rightarrow \mathcal{M}(i,j)$ is induced by a standard embedding (possibly with a dialation).

Lemma

Let $\mathcal{N} = (M_m \oplus M_n \oplus B) *_D C$ and $\mathcal{M} = (M_{n+m} \oplus B) *_D C$, where B, C are semifinite von Neumann algebras and $D = \bigoplus_{i=1}^{K} \overset{p_i^D}{\mathbb{C}}$ with $K \in \mathbb{N} \cup \{\infty\}$. $\mathcal{N}$ is included in $\mathcal{M}$ by including M_m and M_n as blocks on the diagonal of M_{n+m}, and B and C by the identity. Assume there exists a partial isometry in $\mathcal{N}$ between minimal projections in M_m and M_n (for example if there exists a factor $\mathcal{F}$ with $M_m \oplus M_n \subseteq \mathcal{F} \subseteq \mathcal{N}$). Then for any minimal projection $p \in M_m$ such that $p \leq p_i^D$ for some i, $p\mathcal{N}p * L(\mathbb{Z}) \cong p\mathcal{M}p$.

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- Proof uses the partial isometry assumed to reduce it to a lemma proved by Dykema on the amalgamated free product of two by two matrix algebras over $\mathbb{C}^2$
- This shows us the simple steps of the second kind give us standard embedddings, as long as we have the partial isometry necessary.

Example

$$\mathcal{M} = L^\infty(\mu_1) \otimes R *_{\underset{\alpha}{\mathbb{C}} \oplus \underset{1-\alpha}{\mathbb{C}}} L^\infty(\mu_2) \otimes R.$$

$\text{fdim}(\mathcal{M}) = \text{fdim}(A) + \text{fdim}(B) - \text{fdim}(D) = 1+1-(1-\alpha^2-(1-\alpha)^2)$,
Thus $\mathcal{M} = L(F_{1+\alpha^2+(1-\alpha)^2})$.

Theorem (Dykema, R. 2011)

The class $\mathcal{R}_2$ is closed under amalgamated free products over finite dimensional subalgebras. Furthermore
$fdim(A *_D B) = fdim(A) + fdim(B) - fdim(D)$.

Regulated Dimension

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For a von Neumann algebra A with trace τ in $\mathcal{R}_3$ we definite the *Regulated Dimension* of A to be $\text{rdim}(A) = \tau(I_A)^2(\text{fdim}(A) - 1)$.

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- Additive over direct sums
- Invariant over dilation or cutdown by projections with full central support.
- Possibly negative (less than or equal to zero for hyperfinite algebras)
- Does NOT match index of interpolated free group factors.

Regulated Dimension for $\mathcal{R}_4$

- For $A = L(F_t) \otimes B(\mathcal{H})$ we define $\text{rdim}(A) = \text{rdim}(pAp)$ where $p \in A$ is a projection with finite trace

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- For $A = \underset{t}{B(\mathcal{H})}$, we define $\mathrm{rdim}(A) = -t^2$
- For $A = L^\infty(\mu) \otimes B(\mathcal{H}) \otimes R$ and $A = B(\mathcal{H}) \otimes L^\infty(\nu)$ define $\mathrm{rdim}(A) = 0$.

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Thus we can extend rdim to $\mathcal{R}_4$

Semifinite Free Group Factors

We will use the notation $\mathcal{F}_r^t$ to denote the factor which is either $L(F_s)$ or $B(\mathcal{H}) \otimes L(F_s)$, with regulated dimension of r and so that $\tau(I) = t$.

Substandard Embeddings

Definition

Let ϕ be a trace preserving (and not necessarily unital) embedding of $\mathcal{F}_r^t \to \mathcal{F}_{r'}^{t'}$. We say that ϕ is a *substandard embedding* if for some (any) non-zero finite trace projection $p \in \mathcal{F}_r^t$ the embedding $\phi|_p : p\mathcal{F}_r^t p \to \phi(p)\mathcal{F}_{r'}^{t'}\phi(p)$ is standard or an isomorphism.

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Theorem (R. 2012)

Let A and B be semifinite hyperfinite algebras with type I atomic subalgebra D. Then $A *_D B$ is in $\mathcal{R}_4$. If they are finite then the product is in $\mathcal{R}_3$. Furthermore $rdim(A *_D B) = rdim(A) + rdim(B) - rdim(D)$ (where this is defined).

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- Advance k in the same manner as the multimatrix case.

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- Let $\mathcal{M}(i, j, k) = vN(q_k A(i) q_k \cup q_k B(j) q_k)$
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- Advance k in the same manner as the multimatrix case.
- Choose a path so this works.

□

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Proof.

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- Replace induction with inductive limits.

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Example

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