Distributions of Polynomials of Free Semicircular Variables

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July 25, 2013

Introduction

Main Question

Let $X_1, \ldots, X_n$ be freely independent, self-adjoint random variables and let p be a polynomial in n non-commuting variables such that

$$Y := p(X_1, \ldots, X_n)$$

is self-adjoint. What can be said about the spectral distribution μ_Y of Y?

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Belinschi, Mai, and Speicher (2013) developed a method for computing μ_Y. However, as of yet, this technique does not provide direct information about μ_Y.

First Main Theorem

Theorem (Shlyakhtenko, Skoufranis; 2013)

Let $X_1, \ldots, X_n$ *be freely independent, self-adjoint random variables and let* p *be a non-constant polynomial in* n *non-commuting variables such that*

$$Y := p(X_1, \ldots, X_n)$$

is self-adjoint. If the spectral distribution of each X_j *is non-atomic, then the spectral distribution of* Y *is non-atomic.*

Free Entropy

Free Entropy (Voiculescu; 1993)

Let ν be a compactly supported probability measure on $\mathbb{R}$. The free entropy of ν is

$$\Sigma(\nu) := \iint \ln|x-y|\, d\nu(x)\, d\nu(y).$$

It is not difficult to show that if $\Sigma(\nu)$ is finite, then ν is non-atomic.

Upgraded First Main Theorem

Theorem (Shlyakhtenko, Skoufranis; 2013)

Let $X_1, \ldots, X_n$ be freely independent, self-adjoint random variables and let $\{p_{i,j}\}_{i,j=1}^{m}$ be polynomials in n non-commuting variables such that

$$Y := [p_{i,j}(X_1, \ldots, X_n)]_{i,j}$$

is self-adjoint. If the measure of each atom in the spectral distribution of X_j is an integer multiple of $\frac{1}{d_j}$ for some $d_j \in \mathbb{Z}$, then the measure of each atom in the spectral distribution of Y is an integer multiple of $\frac{1}{dm}$ where $d := \prod_{j=1}^{n} d_j$.

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'Self-adjoint' can also be replaced with 'normal' in the above theorem.

Strong Atiyah Property

Definition

Let $\mathcal{A}$ be a $*$-subalgebra of tracial von Neumann algebra $(\mathfrak{M}, \tau)$. We say that $(\mathcal{A}, \tau)$ has the Strong Atiyah Property if for any $n, m \in \mathbb{N}$ and $A \in \mathcal{M}_{m,n}(\mathcal{A})$ the kernel of the induced operator

$$L_A : L_2(\mathfrak{M}, \tau)^{\oplus n} \to L_2(\mathfrak{M}, \tau)^{\oplus m}$$

given by $L_A(\xi) = A\xi$ satisfies $\tau \otimes Id_m(ker(L_A)) \in \mathbb{Z}$.

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Given freely independent, self-adjoint random variables $X_1, \ldots, X_n$, let $(\mathfrak{M}, \tau)$ be the free product von Neumann algebra generated by $X_1, \ldots, X_n$ and let $\mathcal{A}$ be the $*$-subalgebra of all polynomials in $X_1, \ldots, X_n$. If $(\mathcal{A}, \tau)$ has the Strong Atiyah Property and $Y \in \mathcal{A}$ is self-adjoint, then $\mu_Y(\{0\}) = \tau(ker(L_Y)) \in \{0, 1\}$. Thus our main theorem follows.

Strong Atiyah Conjecture

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A discrete group G is said to satisfy the Strong Atiyah Conjecture if $(\mathbb{C}G, \tau_G)$ satisfies the Strong Atiyah Property.

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Note:

- The Strong Atiyah Conjecture does not hold for all groups.
- The free groups $\mathbb{F}_n$ satisfy the Strong Atiyah Conjecture and thus the main theorem follows for freely independent Haar unitaries.

Technical Result

Theorem (Shlyakhtenko, Skoufranis; 2013)

Let $\mathcal{A}$ be a $*$-subalgebra of a tracial von Neumann algebra with separable predual. If $(\mathcal{A}, \tau)$ has the Strong Atiyah Property, then $(\mathcal{A} * \mathbb{C}\mathbb{F}_n, \tau * \tau_{\mathbb{F}_n})$ has the Strong Atiyah Property.

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Let $\mathcal{A}$ be a $*$-subalgebra of a tracial von Neumann algebra with separable predual. If $(\mathcal{A}, \tau)$ has the Strong Atiyah Property, then $(\mathcal{A} * \mathbb{CF}_n, \tau * \tau_{\mathbb{F}_n})$ has the Strong Atiyah Property.

- The proof of the about result was adapted from a result of Schick (2000) which proved the above theorem in the case of discrete groups.
- The main result follows by showing the $*$-algebra $\mathcal{A}$ of all polynomials in n variables acting on $L_2(\mathbb{R}^n, \mu)$ by multiplication for a specific measure μ has the Strong Atiyah Property and the fact that $\mathcal{A} * \mathbb{CF}_n$ then has a freely independent copy of $X_1, \ldots, X_n$.

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This completes the proof of the first result. However, this does not prove we have finite free entropy.

Second Main Theorem

Theorem (Shlyakhtenko, Skoufranis; 2013)

Let $S_1, \ldots, S_n$ be freely independent, semicircular variables and let p be a polynomial in n non-commuting variables such that

$$Y := p(S_1, \ldots, S_n)$$

is self-adjoint. If μ is the spectral distribution of Y, then the Cauchy transform

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z - t} \, d\mu(t)$$

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Recall a power series P is said to be algebraic if there exists an $m \in \mathbb{N}$ and polynomials $\{q_j\}_{j=0}^m$ not all zero such that $\sum_{j=0}^m q_j P^j = 0$.

Algebraic Cauchy Transforms

Note:

- Anderson and Zeitouni (2008) showed that if μ has an algebraic Cauchy transform then there exists a finite subset $A \subseteq \mathbb{R}$ such that μ has a probability density function g such that for each connected interval I in $\mathbb{R} \setminus A$, either $g|_I = 0$ or $g|_I$ is analytic and if $a \in \partial I$ then $g|_I$ decays like $\frac{1}{(z-a)^r}$ when approaching a for some $r \in \mathbb{Q}$.

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- If μ has an algebraic Cauchy transform, it is possible that μ contains atoms.

The hope was to use algebraicity of the Cauchy transform along with no atoms in the distribution to show polynomials of freely independent semicircular variables have finite free entropy.

Upgraded Second Main Theorem

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is algebraic.

Sauer's Technique

When studying geometric group theory, Sauer (2003) proved the algebraicity of the Cauchy transform of polynomials of freely independent Haar unitaries. Let $\mathfrak{M}$ be a finite von Neumann algebra with faithful, normal, tracial state τ. The tracial map on formal power series in one variable is the map $Tr_{\mathfrak{M}} : \mathfrak{M}[[z]] \to \mathbb{C}[[z]]$ defined by

$$Tr_{\mathfrak{M}}\left(\sum_{n\geq 0} T_n z^n\right) = \sum_{n\geq 0} \tau(T_n) z^n.$$

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Lemma (Sauer; 2003)

Let $\mathcal{A}$ be a subalgebra of $\mathfrak{M}$. If

$$Tr_{\mathfrak{M}}(\mathcal{A}_{\mathrm{rat}}[[z]]) \subseteq \mathbb{C}_{\mathrm{alg}}[[z]],$$

then the Cauchy transform G_{μ_A} is algebraic for every positive matrix $A \in \mathcal{M}_\ell(\mathcal{A})$ and any $\ell \in \mathbb{N}$.

Verifying Assumptions of Lemma

In order to verify the assumptions of the above lemma in the case of free Haar unitaries, Sauer used the fact that if $X := \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$ is an alphabet, $W(X)$ denotes the set of all words in X, and $TW(X)$ is the set of all words that reduce to the trivial word, then

$$P_{\text{Haar}} := \sum_{w \in TW(X)} w$$

is an algebraic formal power in $2n$ non-commuting variables.

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is an algebraic formal power in $2n$ non-commuting variables. Analyzing Sauer's proof, to verify our second main theorem it is enough to demonstrate that if $X := \{x_1, \ldots, x_n\}$, then

$$P_{\text{semi}} := \sum_{w \in W(X)} \tau_{\text{semi}}(w(S_1, \ldots, S_n))w$$

is an algebraic formal power in n non-commuting variables.

Schwinger-Dyson Equation

The main tool in demonstrating P_{semi} is algebraic is the Schwinger-Dyson equation (or, more simply, a result from Voiculescu (1998)) that

$$\tau(S_j w(S_1, \ldots, S_n)) = \sum_{\substack{u,v \in W(X) \\ w = u x_j v}} \tau(u(S_1, \ldots, S_n)) \tau(v(S_1, \ldots, S_n)).$$

P_{semi} is Algebraic

$$
\begin{aligned}
& P_{semi} - e \\
= \; & \sum_{j=1}^{n} \sum_{w \in W(X)} \tau(S_j w(S_1, \ldots, S_n)) x_j w
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Hence it is elementary to verify that $P_{semi} - e$ is a solution to the proper algebraic system

$$
z = \sum_{j=1}^{n} x_j z x_j z + x_j^2 z + x_j z x_j + x_j^2.
$$

Open Questions

Some interesting open questions are:

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2. If $X_1, \ldots, X_n$ are freely independent, self-adjoint random variables with spectral distributions with algebraic Cauchy transforms, is the Cauchy transform of the spectral distribution of every self-adjoint polynomial in $X_1, \ldots, X_n$ algebraic?

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2. If $X_1, \ldots, X_n$ are freely independent, self-adjoint random variables with spectral distributions with algebraic Cauchy transforms, is the Cauchy transform of the spectral distribution of every self-adjoint polynomial in $X_1, \ldots, X_n$ algebraic?
3. Are there more analytic techniques to proving these results? In particular, can the results of Belinschi, Mai, and Speicher (2013) be applied to obtain these results?

Thanks for Listening!