Operator-valued free probability theory and the self-adjoint linearization trick

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Workshop on Analytic, Stochastic, and Operator Algebraic Aspects of Noncommutative Distributions and Free Probability

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Contents

Definition

Let $p \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$ be given. A matrix

$$L_p := \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in \mathrm{M}_N(\mathbb{C}\langle X_1, \ldots, X_n \rangle),$$

of dimension $N \in \mathbb{N}$, where

- u and v are row and column vectors, respectively, both of dimension $N-1$ with entries in $\mathbb{C}\langle X_1, \ldots, X_n \rangle$ and
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is called **a linearization of** p, if the following conditions are satisfied:

(i) There are matrices $b_0, \dots, b_n \in \mathrm{M}_N(\mathbb{C})$, such that

$$L_p = b_0 \otimes 1 + b_1 \otimes X_1 + \cdots + b_n \otimes X_n \in \mathrm{M}_N(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle.$$

(ii) It holds true that $p = -uQ^{-1}v$.

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Let $\mathcal{A}$ be a complex unital algebra and let $x_1, \dots, x_n \in \mathcal{A}$ be given. Put $P := p(x_1, \dots, x_n)$ and

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By using the notation $\Lambda(z) := \mathrm{diag}(z, 0, \dots, 0) \in \mathrm{M}_N(\mathbb{C})$, we get

$z - P$ is invertible in $\mathcal{A}$ $\iff$ $\Lambda(z) - L_P$ is invertible in $\mathrm{M}_N(\mathbb{C}) \otimes \mathcal{A}$

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(ii) Any polynomial $p \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ has a linearization L_p. If p is self-adjoint, L_p can be chosen to be self-adjoint as well.

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Definition

An **operator-valued** C^***-probability space** $(\mathcal{A}, E, \mathcal{B})$ consists of

- a unital C^*-algebra $\mathcal{A}$,
- a unital C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ and
- a **conditional expectation** $E : \mathcal{A} \to \mathcal{B}$, i.e. a positive and unital map satisfying
  - $E[b] = b$ for all $b \in \mathcal{B}$ and
  - $E[b_1 a b_2] = b_1 E[a] b_2$ for all $a \in \mathcal{A}$, $b_1, b_2 \in \mathcal{A}$.

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Example

Let $(\mathcal{A}_0, \phi)$ be a C^*-probability space. Then

$$\mathcal{A} := \mathrm{M}_N(\mathbb{C}) \otimes \mathcal{A}_0, \qquad \mathcal{B} := \mathrm{M}_N(\mathbb{C}) \qquad \text{and} \qquad E := \mathrm{id}_{\mathrm{M}_N(\mathbb{C})} \otimes \phi$$

gives an operator-valued C^*-probability space $(\mathcal{A}, E, \mathcal{B})$.

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$$\mathbb{H}^{\pm}(\mathcal{B}) := \{b \in \mathcal{B} |\ \exists \varepsilon > 0 :\ \pm\Im(b) \geq \varepsilon 1\}$$

the **upper and lower half-plane**, respectively, where we use the notation

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- the h-transform $h_x :\ \mathbb{H}^+(\mathcal{B}) \to \overline{\mathbb{H}^+(\mathcal{B})},\ h_x(b) := F_x(b) - b$.

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Let $(\mathcal{A}, \phi)$ be a C^*-probability space and let $x_1, \ldots, x_n \in \mathcal{A}$ be self-adjoint and freely independent.

Question

Given a self-adjoint non-commutative polynomial $p \in \mathbb{C}\langle X_1, \ldots, X_n \rangle$.
How can we calculate the distribution of $p(x_1, \ldots, x_n)$ out of the given distributions of $x_1, \ldots, x_n$?

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Solution (Belinschi, M., Speicher, 2013)

Combine the linearization trick in its self-adjoint version by Anderson with results about the operator-valued free additive convolution in the setting of operator-valued C^*-probability spaces.

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This gives an algorithmic solution for the question above, which is moreover easily accessible for numerical computations!

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Anderson's linearization trick shows that there is an self-adjoint operator

$$L_P := b_0 \otimes 1 + b_1 \otimes x_1 + \cdots + b_n \otimes x_n \in \mathrm{M}_N(\mathbb{C}) \otimes \mathcal{A},$$

such that we have with respect to $E = \mathrm{id}_{\mathrm{M}_N(\mathbb{C})} \otimes \phi$ for all $z \in \mathbb{C}^+$

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where $\Lambda_\varepsilon(z) := \mathrm{diag}(z, i\varepsilon, \dots, i\varepsilon) \in \mathbb{H}^+(\mathrm{M}_N(\mathbb{C}))$.

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Observation

$b_1 \otimes x_1, \dots, b_n \otimes x_n$ are free with amalgamation over $\mathrm{M}_N(\mathbb{C})$.

Theorem (Belinschi, M., Speicher, 2013)

Assume that $(\mathcal{A}, E, \mathcal{B})$ is an operator-valued C^*-probability space.

If $x, y \in \mathcal{A}$ are free with respect to E, then there exists a unique pair of (Fréchet-)holomorphic maps

$$\omega_1, \omega_2 : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$$

such that

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Moreover, ω_1 and ω_2 can easily be calculated via the following **fixed point iterations** on $\mathbb{H}^+(\mathcal{B})$:

$$\begin{array}{rcll} w & \mapsto & h_y(b + h_x(w)) + b & \text{for } \omega_1(b) \\ w & \mapsto & h_x(b + h_y(w)) + b & \text{for } \omega_2(b) \end{array}$$

Appetizer (coming from free stochastic integrals)

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Consider

$$p(X_1, X_2) = \frac{1}{2}(X_1^3 + X_1X_2X_1 + X_2X_1X_2 + X_2^3) - (X_1 + X_2).$$

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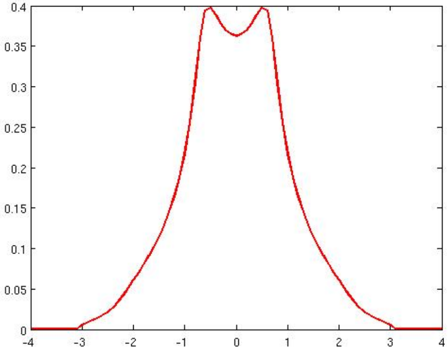
The density of the distribution of $p(s_1, s_2)$ looks like:



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where $(s_1, \dots, s_d)$ is a semicircular family.

$\Rightarrow$ $p(\sigma_{1,N}, \dots, \sigma_{d,N}) \xrightarrow[N\to\infty]{\text{dist}} p(s_1, \dots, s_d)$,
for any (self-adjoint) polynomial $p \in \mathbb{C}\langle X_1, \dots, X_d\rangle$.

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Then $(X_n)_{n \in \mathbb{N}}$ are self-adjoint elements in $\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{A}$, which are identically distributed and free with respect to $E = \mathrm{id}_{\mathrm{M}_k(\mathbb{C})} \otimes \phi$ with $E[X_n] = 0$ for all $n \in \mathbb{N}$.

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Then $(X_n)_{n \in \mathbb{N}}$ are self-adjoint elements in $\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{A}$, which are identically distributed and free with respect to $E = \mathrm{id}_{\mathrm{M}_k(\mathbb{C})} \otimes \phi$ with $E[X_n] = 0$ for all $n \in \mathbb{N}$. Hence,

$$\Sigma_N := \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n$$

converges as $N \to \infty$ in distribution (with respect to E) to the operator-valued semicircular element

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Note that we have

$$L_p(s_1, \ldots, s_n) = b_0 \otimes 1 + S \qquad \text{and} \qquad L_p(\sigma_{1,N}, \ldots, \sigma_{d,N}) = b_0 \otimes 1 + \Sigma_N.$$

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Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued C^*-probability space with faithful conditional expectation E and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of identically distributed and self-adjoint elements in $\mathcal{A}$, satisfying $E[X_n] = 0$, which are free with respect to E. Then

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We use the notation $m_n^X(b_1, \dots, b_{n-1}) := E[Xb_1X \dots Xb_{n-1}X]$ and

$$\|m_n^X\| := \sup_{\|b_1\|\leq 1, \dots, \|b_{n-1}\|\leq 1} \|m_n^X(b_1, \dots, b_{n-1})\| \leq \|X\|^n.$$

Back to the multivariate case

Back to the multivariate case

Corollary

For each self-adjoint $p \in \mathbb{C}\langle X_1, \dots, X_d \rangle$, there are $M, C > 0$ such that

$$|G_{p(\sigma_{1,N},\dots,\sigma_{d,N})}(z) - G_{p(s_1,\dots,s_d)}(z)| \leq N^{-\frac{1}{10}} \left(M + \frac{C}{\Im(z)^2} \right)$$

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