Multiplicative geometric structures

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(joint with Thiago Drummond, UFRJ)

Workshop on EDS and Lie theory

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Outline:

1. Motivation: geometry on Lie groupoids
2. Multiplicative structures
3. Infinitesimal/global correspondence
4. Examples and applications

1. Motivation

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Recurrent problem: infinitesimal counterparts, integration...

Some cases have been considered, through different methods...

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[1] Crainic: Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes, *Comment. Math. Helv.* (2003)

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2. Multiplicative tensors

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View $\tau \in \Gamma(\wedge^q T\mathcal{G} \otimes \wedge^p T^*\mathcal{G})$ as *function* $\bar{\tau} \in C^\infty(\mathbb{G})$:

$$(\alpha_1, \ldots, \alpha_q, X_1, \ldots, X_p) \overset{\bar{\tau}}{\mapsto} \tau(\alpha_1, \ldots, \alpha_q, X_1, \ldots, X_p).$$

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Definition:

τ is **multiplicative** if $\bar{\tau} \in C^\infty(\mathbb{G})$ is multiplicative. (1-cocycle)

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Enough to use particular types of sections $\mathbb{a} \in \Gamma(\mathbb{A})$...

Key facts:

◇ Information about $\mathcal{L}_{\mathbb{a}^r}\bar{\tau}$ encoded in

$$\mathcal{L}_{a^r}\tau \in \Gamma(\wedge^q T\mathcal{G} \otimes \wedge^p T^*\mathcal{G}),$$

$$i_{a^r}\tau \in \Gamma(\wedge^q T\mathcal{G} \otimes \wedge^{p-1} T^*\mathcal{G}),$$

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◇ The map $\mathbb{t}^* : C^\infty(\mathbb{M}) \to C^\infty(\mathbb{G})$ restricts to

$$\Gamma(\wedge^q A \otimes \wedge^p T^*M) \to \Gamma(\wedge^q T\mathcal{G} \otimes \wedge^p T^*\mathcal{G}),$$
$$\chi \otimes \alpha \mapsto \chi^r \otimes t^*\alpha$$

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$$\mathcal{L}_{a^r}\tau = \mathbb{t}^*(D(a)),$$
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How about cocycle equations?

Cocycle equations for (D, r, l):

(1) $D([a,b]) = a.D(b) - b.D(a)$

(2) $l([a,b]) = a.l(b) - i_{\rho(b)}D(a)$

(3) $r(\mathcal{L}_{\rho(a)}\alpha) = a.r(\alpha) + i_{\rho^*(\alpha)}D(a)$

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(Redundancies...)

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Theorem: (B., Drummond)
1-1 correspondence between $\tau \in \Gamma(\wedge^q T\mathcal{G} \otimes \wedge^p T^*\mathcal{G})$ multiplicative and (D, l, r), where

$$D : \Gamma(A) \to \Gamma(\wedge^q A \otimes \wedge^p T^*M), \quad \text{Leibniz-like condition,}$$

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(more general tensors, coefficients in reps...)

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Infinitesimal components become:

$$\delta : \Gamma(\wedge^\bullet A) \to \Gamma(\wedge^{\bullet+q-1} A),$$

such that

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E.g. (quasi-)Poisson groupoids and (quasi-)Lie bialgebroids...

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E.g. symplectic groupoids and Poisson structures...

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GLA relative to *Frolicher-Nijenhuis* bracket on multiplicative $\Omega^\bullet(\mathcal{G}, T\mathcal{G})$...

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complex Lie group: Lie bracket complex bilinear

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complex Lie group: Lie bracket complex bilinear

holomorphic vector bundle: flat, partial T^{10}-connection

Particular case $p = 1$: $J : T\mathcal{G} \to T\mathcal{G}$

Components are (D, J_A, J_M),

$J_A : A \to A$,

$J_M : TM \to TM$,

$D : \Gamma(A) \to \Gamma(T^*M \otimes A)$.

Can analyze $\frac{1}{2}[J, J] = N_J \in \Omega^2(\mathcal{G}, T\mathcal{G})$ infinitesimally...

E.g. holomorphic Lie groupoids $\rightleftharpoons$ holomorphic Lie algebroids...

complex Lie group: Lie bracket complex bilinear

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more: holomorphic symplectic/Poisson groupoids, almost product...

Thank you