On the Van Est homomorphism for Lie groupoids

Eckhard Meinrenken (based on joint work with David Li-Bland)

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Overview

Let $G \rightrightarrows M$ be a Lie groupoid, with Lie algebroid $A = \mathrm{Lie}(G)$

Weinstein-Xu (1991) constructed a cochain map

$$\mathrm{VE}\colon \mathrm{C}^\bullet(G) \to \mathrm{C}^\bullet(A)$$

from smooth groupoid cochains to the Chevalley-Eilenberg complex of the Lie algebroid A of G.

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Crainic (2003) proved a *Van Est Theorem* for this map.

Weinstein, Mehta (2006), and Abad-Crainic (2008, 2011) generalized to

$$\mathrm{VE}\colon \mathrm{W}^{\bullet,\bullet}(G) \to \mathrm{W}^{\bullet,\bullet}(A)$$

for suitably defined *Weil algebras*.

Overview

Applications

- Foliation theory
- Integration of (quasi-)Poisson manifolds and Dirac structures
- Multiplicative forms on groupoids (Mackenzie-Xu, Bursztyn-Cabrera-Ortiz)
- Index theory (Posthuma-Pflaum-Tang)
- Lie pseudogroups and Spencer operators (Crainic-Salazar-Struchiner),
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Using the Fundamental Lemma of homological perturbation theory, we'll give a simple construction of VE (and its properties).

Lie groupoid cohomology

Let $G \rightrightarrows M$ be a Lie groupoid over $M \subseteq G$.

$$m_0 \xleftarrow{g} m_1.$$

Multiplication $(g_1, g_2) \mapsto g_1 g_2$ defined for *composable arrows*:

$$\left(m_0 \xleftarrow{g_1} m_1 \xleftarrow{g_2} m_2\right) \;\mapsto\; \left(m_0 \xleftarrow{g_1 g_2} m_2\right).$$

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Examples

- Lie group $G \rightrightarrows \mathrm{pt}$
- Pair groupoid $\mathrm{Pair}(M) = M \times M \rightrightarrows M$
- Fundamental groupoid $\Pi(M) \rightrightarrows M$
- Foliation groupoid(s), e.g., $\Pi_{\mathcal{F}}(M) \rightrightarrows M$
- Gauge groupoids of principal bundles
- Action groupoids $K \ltimes M \rightrightarrows M$
- Groupoids associated with hypersurfaces

Lie groupoid cohomology

Let B_pG be the manifold of p-arrows $(g_1, \ldots, g_p)$:

$$m_0 \xleftarrow{g_1} m_1 \xleftarrow{g_2} m_2 \cdots \xleftarrow{g_p} m_p$$

It is a simplicial manifold, with face maps

$$\partial_i\colon B_pG \to B_{p-1}G,\ \ i = 0, \ldots, p$$

removing m_i and degeneracies $\epsilon_i\colon B_pG \to B_{p+1}G$ repeating m_i.

For example

$$\partial_1\Big(m_0 \xleftarrow{g_1} m_1 \xleftarrow{g_2} m_2 \cdots \xleftarrow{g_p} m_p\Big) = \Big(m_0 \xleftarrow{g_1 g_2} m_2 \cdots \xleftarrow{g_p} m_p\Big).$$

Lie groupoid cohomology

Groupoid cochain complex: $\mathrm{C}^\bullet(G) := C^\infty(B_\bullet G)$ with differential

$$\delta = \sum_{i=0}^{p+1} (-1)^i \partial_i^* \colon C^\infty(B_p G) \to C^\infty(B_{p+1} G)$$

and algebra structure $\mathrm{C}^p(G) \otimes \mathrm{C}^{p'}(G) \to \mathrm{C}^{p+p'}(G)$,

$$f \cup f' = \mathrm{pr}^* f \ (\mathrm{pr}')^* f',$$

where pr, pr′ are the 'front face' and 'back face' projections.

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Variations:

- Normalized subcomplex $\widetilde{\mathrm{C}}^\bullet(G)$: kernel of degeneracy maps ϵ_i.
- More generally, with coefficients in G-modules $S \to M$.
- $\mathrm{C}^\bullet(G)_M := C^\infty(B_\bullet G)_M$, the germs along $M \subseteq B_p G$.
- Extends to double complex $\mathrm{W}^{\bullet,\bullet}(G) := \Omega^\bullet(B_\bullet G)$.

Lie groupoid cohomology

Example (Alexander-Spanier complex)

$\mathrm{C}^\bullet(\mathrm{Pair}(M))_M = C^\infty(M^{p+1})_M$

$$(\delta f)(m_0,\ldots,m_{p+1}) = \sum_{i=0}^{p+1}(-1)^i f(m_0,\ldots,\widehat{m_i},\ldots,m_{p+1})$$

Lie algebroid cohomology

Let $A \to M$ be a Lie algebroid, with anchor $\mathsf{a}\colon A \to TM$ and bracket $[\cdot,\cdot]_A$ on $\Gamma(A)$. Thus

$$[X, fY] = f[X, Y] + (\mathsf{a}(X)f)\, Y.$$

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Examples

- Lie algebra $\mathfrak{g}$
- Tangent bundle TM
- Tangent bundle to foliation $T_{\mathcal{F}}M \subset TM$
- Atiyah algebroid of principal bundle
- Cotangent Lie algebroid of Poisson manifold
- Action Lie algebroids $\mathfrak{k} \ltimes M$
- Lie algebroids associated with hypersurfaces
- ...

Lie algebroid cohomology

The Chevalley-Eilenberg complex is $C^\bullet(A) = \Gamma(\wedge^\bullet A^*)$ with differential

$$\begin{aligned}(\mathrm{d}_{CE}\phi)&(X_0,\ldots,X_p)\\ =&\sum_{i=0}^{p}(-1)^i \mathsf{a}(X_i)\phi(X_0,\ldots,\widehat{X}_i,\ldots,X_p)\\ &+\sum_{i<j}(-1)^{i+j}\phi([X_i,X_j],X_0,\ldots,\widehat{X}_i,\ldots,\widehat{X}_j,\ldots,X_p)\end{aligned}$$

and with product the wedge product.

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and with product the wedge product.

- More generally, with coefficients in A-modules $S \to M$.
- Extends to double complex $W^{\bullet,\bullet}(A)$
- For $A = T_{\mathcal{F}}M$, get foliated de Rham complex $\Omega_{\mathcal{F}}(M)$.

From Lie groupoids to Lie algebroids

A Lie groupoid $G \rightrightarrows M$ has an associated Lie algebroid:

- $\mathrm{Lie}(G) = \nu(M, G)$
- anchor $\mathsf{a}\colon \mathrm{Lie}(G) \to TM$ induced from $Tt - Ts\colon TG \to TM$,
- $[\cdot, \cdot]$ from $\Gamma(\mathrm{Lie}(G)) = \mathrm{Lie}(\Gamma(G))$ where $\Gamma(G)$ is the group of bisections.

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We'll explain this map using a double complex.

Van Est double complex

Define a principal G-bundle

$$\begin{array}{ccc} E_pG & \xrightarrow[\kappa_p]{} & B_pG \\ {\scriptstyle \pi_p}\downarrow & & \\ M & & \end{array}$$

where $E_pG \subseteq G^{p+1}$ consists of elements $(a_0, \ldots, a_p)$ with common source:

$$\begin{array}{ccccccc} & & & m & & & \\ & \swarrow^{a_0} & & \downarrow{\scriptstyle a_1} & \searrow^{\cdots} & & \searrow^{a_p} \\ m_0 & & & m_1 & & \ldots & & m_p \end{array}$$

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.. and where π_p and κ_p take such an element

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to the common source m, respectively to

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m_0 \xleftarrow{\;g_1\;} m_1 \xleftarrow{\;g_2\;} \ldots \xleftarrow{\;g_p\;} m_p
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with $g_i = a_i a_{i-1}^{-1}$.

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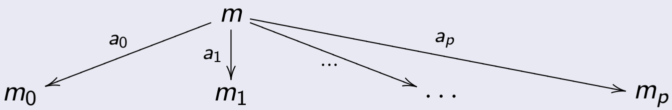

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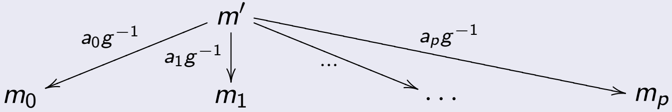
$$m_0 \xleftarrow{g_1} m_1 \xleftarrow{g_2} \cdots \xleftarrow{g_p} m_p$$

with $g_i = a_i a_{i-1}^{-1}$. The groupoid action of an element $m' \xleftarrow{g} m$ takes this element to



Van Est double complex

View M as a simplicial manifold (with all $M_p = M$). Then

$$\begin{array}{ccc} E_pG & \xrightarrow[\kappa_p]{} & B_pG \\ {\scriptstyle \pi_p}\downarrow & & \\ M & & \end{array}$$

is a *simplicial* principal G-bundle. The map

$$\iota_p \colon M \to E_pG, \ m \mapsto (m, \ldots, m)$$

is a simplicial inclusion; $\pi_p \circ \iota_p = \mathrm{id}$.

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Theorem

There is a (canonical) simplicial deformation retraction from $E_\bullet G$ onto M.

See: G. Segal, Classifying spaces and spectral sequences (1968).

Van Est double complex

Since $E_\bullet G$ is a simplicial manifold, have cochain complex

$$C^\infty(E_\bullet G), \quad \delta = \sum_{i=0}^{p+1} (-1)^i \partial_i^*,$$

with cochain maps

$$\begin{array}{ccc} C^\infty(E_\bullet G) & \xleftarrow{\kappa^*} & C^\infty(B_\bullet G) = \mathrm{C}^\bullet(G) \\ \uparrow{\scriptstyle \pi^*} & & \\ C^\infty(M_\bullet) & & \end{array}$$

The map $h\colon C^\infty(E_{p+1}G) \to C^\infty(E_p G)$,

$$(hf)(a_0, \ldots, a_p) = \sum_{i=0}^{p} (-1)^{i+1} f(a_0, \ldots, a_i, m \ldots, m)$$

with $m = \pi_p(a_0, \ldots, a_p)$, is a δ-homotopy: $h\delta + \delta h = 1 - \pi^* \iota^*$.

Van Est double complex

Let $A = \mathrm{Lie}(G)$. Since $\kappa_p \colon E_pG \to B_pG$ is a principal G-bundle, have

$$T_{\mathcal{F}} E_p G \cong \pi_p^* A,$$

and $T_{\mathcal{F}} E_\bullet G \to A$ is a morphism of simplicial Lie algebroids.

Get double complex

$$(\Omega^q_{\mathcal{F}}(E_pG),\ \delta,\ \mathrm{d})$$

with $\mathrm{d} = (-1)^p \mathrm{d}_{CE}$.

Van Est double complex

Have morphism of double complexes

$$\begin{array}{ccc} \Omega^\bullet_{\mathcal{F}}(E_\bullet G) & \xleftarrow{\ \kappa^*\ } & C^\infty(B_\bullet G) = \mathsf{C}^\bullet(G) \\ \Big\uparrow{\scriptstyle \pi^*} & & \\ \Gamma(\wedge^\bullet A^*_\bullet) = \mathsf{C}^\bullet(A_\bullet) & & \end{array}$$

where $\mathrm{d} = 0$ on $C^\infty(B_\bullet G)$.

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Want to turn this into homotopy equivalence with respect to $\mathrm{d} + \delta$.

Perturbation theory

Set-up:

- $(C^{\bullet,\bullet}, \mathrm{d}, \delta)$ be a double complex
- $i\colon D \hookrightarrow C$ a sub-double complex
- $r\colon C \to D$ a (bigraded) projection
- $h\colon C^{\bullet,\bullet} \to C^{\bullet-1,\bullet}$ with $h\delta + \delta h = 1 - i \circ r$.

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Lemma (Fundamental Lemma of homological perturbation theory)

Put

$$i' = (1 + h\mathrm{d})^{-1} i,\ \ r' = r(1 + \mathrm{d}h)^{-1},\ \ h' = h(1 + \mathrm{d}h)^{-1}.$$

Then $i' \circ r'$ is a cochain map for $\mathrm{d} + \delta$, and

$$h'(\mathrm{d} + \delta) + (\mathrm{d} + \delta)h' = 1 - i' \circ r'.$$

References: Gugenheim-Lambe-Stasheff, Brown, Crainic, ...

Van Est map

In our case, this shows that

$$\iota^* \circ (1 + \mathrm{d}h)^{-1} \colon \Omega^\bullet_{\mathcal{F}}(E_\bullet G) \to \Gamma(\wedge^\bullet A^*_\bullet)$$

is a homotopy equivalence, with homotopy inverse $(1 + h\mathrm{d})^{-1} \circ \pi^*$.

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$$\Gamma(\wedge^\bullet A^*_\bullet) \rightleftarrows \Gamma(\wedge^\bullet A^*)$$

Hence:

Theorem

The map

$$\iota_0^* \circ (1 + \mathrm{d}h)^{-1} \colon \Omega_{\mathcal{F}}(EG) \to \Gamma(\wedge A^*)$$

is a homotopy equivalence (for $\mathrm{d} + \delta$*), with homotopy inverse* π_0^**.*

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is a homotopy equivalence (for $\mathrm{d} + \delta$), with homotopy inverse π_0^*.

Using $\kappa^* \colon C^\infty(BG) \to \Omega_{\mathcal{F}}(EG)$ we get the desired cochain map:

Van Est map

Definition

The composition

$$\mathsf{VE} := \iota_0^* \circ (1 + \mathrm{d}h)^{-1} \circ \kappa^* \colon C^\infty(BG) \to \Gamma(\wedge A^*)$$

is called the Van Est map.

Proposition

This map agrees with the Van Est map of Weinstein-Xu.

Van Est map

Equivalently, we may write $\mathsf{VE} = \iota_0^* \circ (1 + \mathrm{d}h)^{-1} \circ \kappa^*$ as

$$\mathsf{VE} = (-1)^p \iota_0^* \circ (\mathrm{d} \circ h)^p \circ \kappa_p^* \colon C^\infty(B_p G) \to \Gamma(\wedge^p A^*)$$

corresponding to a 'zig-zag': E.g., for $p = 2$

$$\begin{array}{ccccccc}
C^\infty(B_2 G) & \xrightarrow[\kappa_2^*]{} & \Omega^0_{\mathcal{F}}(E_2 G) & & & & \\
& & \Big\downarrow h & & & & \\
& & \Omega^0_{\mathcal{F}}(E_1 G) & \xrightarrow[\mathrm{d}]{} & \Omega^1_{\mathcal{F}}(E_1 G) & & \\
& & & & \Big\downarrow h & & \\
& & & & \Omega^1_{\mathcal{F}}(E_0 G) & \xrightarrow[\mathrm{d}]{} & \Omega^2_{\mathcal{F}}(E_0 G) \\
& & & & & & \Big\downarrow \iota_2^* \\
& & & & & & \Gamma(\wedge^2 A^*).
\end{array}$$

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$\rightsquigarrow$ homotopy operator $k \colon \Omega^\bullet_{\mathcal{F}}(E_\bullet G) \to \Omega^{\bullet-1}_{\mathcal{F}}(E_\bullet G)$ with $k\mathrm{d} + \mathrm{d}k = 1 - \kappa^* j^*$.

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- $\rightsquigarrow$ 'integration' $j^* \circ (1 + \delta k)^{-1} \circ \pi^* \colon \Gamma(\wedge^\bullet A^*) \to C^\infty(B_\bullet G)$.

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$\rightsquigarrow$ 'integration' $j^* \circ (1 + \delta k)^{-1} \circ \pi^* \colon \Gamma(\wedge^\bullet A^*) \to C^\infty(B_\bullet G)$.

Recall that $C^\infty(BG)_M$ denotes 'germs'.

Corollary

For any (local) Lie groupoid,

$$\mathrm{VE} \colon C^\infty(BG)_M \to \Gamma(\wedge A^*)$$

is a quasi-isomorphism.

Product structure

Consider following situation:

- (C, d, δ) be a bigraded bidifferential algebra
- $i\colon D \hookrightarrow C$ a sub-bidifferential algebra
- $r\colon C \to D$ a projection preserving products
- $h\colon C^{\bullet,\bullet} \to C^{\bullet-1,\bullet}$ with

$$h\delta + \delta h = 1 - i \circ r.$$

Lemma (Gugenheim-Lambe-Stasheff)

Suppose h is a twisted derivation

$$h(\omega \cup \omega') = h(\omega) \cup (i \circ r)(\omega') + (-1)^{|\omega|}\omega \cup h(\omega'),$$

and that it satisfies the side conditions $h \circ h = 0$ and $h \circ i = 0$. Then

$$i' = (1 + h\mathrm{d})^{-1} i, \;\; r' = r(1 + \mathrm{d}h)^{-1}$$

are morphisms of graded differential algebras (w.r.t. $\mathrm{d} + \delta$).

Product structure

In our case, these conditions hold once we restrict to the normalized subcomplex

$$\widetilde{C}^\infty(B_\bullet G) \subset C^\infty(B_\bullet G)$$

(i.e. kernel of the degeneracy maps ϵ_i^*). Hence we obtain

The map

$$\mathrm{VE}\colon \widetilde{C}^\infty(BG) \to \Gamma(\wedge A^*)$$

preserves products.

Van Est map

The discussion also applies to the more general Van Est map

$$\mathrm{VE}\colon \mathrm{W}^{p,q}(G) = \Omega^q(B_pG) \to \mathrm{W}^{p,q}(A).$$

In particular:

- given a retraction of G along t-fibers there is a canonical 'integration map' in opposite direction
- over the normalized complex, VE preserves products

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Definition of the Weil algebra $W(A)$ of a Lie algebroid: See Weinstein, Mehta (2008) or Abad-Crainic (2012).

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Definition

$W^{p,q}(A)$ are the skew-symmetric multi-linear q-forms on $A \times_M A \cdots \times_M A$ (p factors).

Van Est map

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- $R := \wedge A^*$
- $\mathfrak{X}^1_R = \mathfrak{der}(R)$. I.e., $\Gamma(\mathfrak{X}^1_R) = \mathfrak{der}(\Gamma(R))$.

Van Est map

Another definition: 'Kähler differentials'. Start with any vector bundle $A \to M$.

- $R := \wedge A^*$
- $\mathfrak{X}^1_R = \mathfrak{der}(R)$. I.e., $\Gamma(\mathfrak{X}^1_R) = \mathfrak{der}(\Gamma(R))$.
- $\Omega^1_R = \hom_R(\mathfrak{X}^1_R, R)$.

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- $\Omega_R^1 = \hom_R(\mathfrak{X}_R^1, R)$.
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Any degree k derivation X of $\Gamma(R)$ extends to a degree $(k,0)$ derivation $\mathcal{L}_X$ of $\mathrm{W}(A)$. If A is a Lie algebroid, apply this to $X = \mathrm{d}_{CE}$.

Thanks.